The Influence of Wing Flexibility on the Longitudinal Dynamics of a Flapping Wing Micro Air Vehicle in Hover

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Many studies in recent years have analyzed the stability and control of flapping wing micro air vehicles (FWMAVs) using computational, experimental and analytical methods that assume rigid wings. However, wing flexibility is clearly observed in natural flyers such as birds, bats, and insects. It is important to investigate the extent to which this flexibility affects the stability and controllability of FWMAVs. In this study, we develop the equations of motion of a FWMAV and analyze the longitudinal stability of a FWMAV in hover considering both rigid wings and flexible wings. We then solve the equations using biomimetic morphological parameters and wing kinematics at the bumblebee scale. Analysis consists of both rigid wing modeling based on quasi-steady aerodynamics as well as fully coupled Navier-Stokes-Euler-Bernoulli solutions to the two-dimensional lift produced by a rigid and flexible wing using the same kinematics. The system is perturbed from equilibrium and the open-loop stability is assessed. Comparisons between the behavior of the vehicle with flexible wings and the vehicle with rigid wings demonstrate that there are noticeable differences in the longitudinal response of the dynamics of a flyer with flexible wings compared to rigid wings.

Nomenclature

\[\begin{align*}
\text{AoA} &= \text{local aerodynamic angle of attack} \\
C_L &= \text{coefficient of lift} \\
C_D &= \text{coefficient of drag} \\
E &= \text{Young’s modulus} \\
F &= \text{fluid force acting on the wing} \\
I &= \text{mass moment of inertia} \\
M &= \text{moment} \\
Re &= \text{Reynolds number, } Uc / \nu \\
U &= \text{maximum plunge velocity of wing during a full stroke} \\
W &= \text{weight} \\
\dddot{a} &= \text{acceleration vector} \\
b &= \text{wing semi-span (from wing root to wing tip)} \\
c &= \text{chord} \\
f &= \text{wing flapping frequency} \\
f_1 &= \text{first natural frequency of the wing} \\
g &= \text{gravitational acceleration} \\
h_s &= \text{thickness of the wing} \\
k &= \text{reduced frequency, } \pi fc / U \\
m &= \text{mass}
\end{align*}\]

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\[ p \quad = \quad \text{pressure} \quad [\text{Pa}] \]
\[ \mathbf{r} \quad = \quad \text{position vector} \quad [\text{m}] \]
\[ t \quad = \quad \text{time; wing thickness} \quad [\text{s}]; [\text{m}] \]
\[ u, v, w \quad = \quad \text{x-, y-, and z-components of velocity (body frame)} \quad [\text{m/s}] \]
\[ \mathbf{v} \quad = \quad \text{velocity vector} \quad [\text{m/s}] \]
\[ x, y, z \quad = \quad \text{translational coordinates of the body} \quad [\text{m}] \]
\[ \Lambda \quad = \quad \text{pitch amplitude} \quad [\text{°}] \]
\[ Z \quad = \quad \text{stroke amplitude} \quad [\text{°}] \]
\[ \alpha \quad = \quad \text{pitch of wing relative to the stroke plane} \quad [\text{°}] \]
\[ \beta \quad = \quad \text{stroke plane angle} \quad [\text{°}] \]
\[ \zeta \quad = \quad \text{stroke or flapping angle of the wing} \quad [\text{rad}] \]
\[ \omega \quad = \quad \text{angular velocity} \quad [\text{rad/s}] \]
\[ \nu \quad = \quad \text{kinematic viscosity of the fluid} \quad [\text{m}^2/\text{s}] \]
\[ \rho_f \quad = \quad \text{density of the fluid} \quad [\text{kg/m}^3] \]
\[ \rho_w \quad = \quad \text{density of the wing structure} \quad [\text{kg/m}^3] \]

I. Introduction

The study of the aeromechanics of a biological flyer or flapping wing micro air vehicle (FWMAV) is a complex endeavor. The design of such small vehicles presents many well-known challenges including aerodynamics, flight dynamics, power storage and transmission, sensory mechanisms, and control among others. In each field, significant progress has been made in recent years, but an integrated framework for modeling and designing these complex systems remains elusive.

A host of research has been performed in order to understand the complex aerodynamics at this flow regime where the Reynolds number is \( Re = O(10^2)-O(10^4) \). Many stability studies rely on the quasi-steady aerodynamic models produced by Sane and Dickinson\(^2\)-\(^4\), who offered a model which relied partly on the quasi-steady work of Theodorsen\(^5\) and partly on their own experiments using a dynamically-scaled model rigid wing in an oil tank. They published a quasi-steady model which is efficient for use in stability and control analysis because it has low computational cost. Other studies have also demonstrated the various unsteady physical mechanisms of lift production at these low Reynolds numbers, such as clap and fling\(^6\), prolonged attachment of leading-edge vortices\(^7\), wake-capture\(^2\), and rapid wing rotation\(^2\) and have indicated the means by which this lift and thrust force can be made in the most efficient manner possible\(^8\)-\(^10\). As interest in this field has grown, several researchers have pursued computational solutions to the Navier-Stokes equations\(^11\),\(^12\). Recently, several studies\(^13\)-\(^15\) have demonstrated the importance of considering wing flexibility in aerodynamic analysis, and its significant impact on the aerodynamic force generation mechanisms, efficiency, and the timing of passive wing kinematics. Each of these findings appears to have ramifications in the realm of flight dynamics as well. Indeed aerodynamic analysis in general, though an important prerequisite to successful design, must naturally be proceeded by an analysis of stability and control of the flyer. Stability analysis seeks to address the tendency of a FWMAV in equilibrium to stay in equilibrium, while control studies seek to develop and implement control strategies to ensure the actual motion of the system matches the desired motion.

Many investigators have studied the stability of FWMAVs in recent years, which have been summarized by Sun\(^16\) and by Taha, Hajj, and Nayfeh\(^17\). Several researchers have conducted experiments on live insects: Taylor and Thomas\(^18\) directly studied the motion of the desert locust, and Ristroph et al.\(^19\) concentrated on the stability of fruit flies. Cheng, Deng, and Hedrick\(^20\) focused on the control of a hawk moth in hover. More recently Ortega-Jimenez, Mittal and Hedrick\(^21\) investigated the hawkmoth stability in whirlwind vortices. There have been several efforts at numerical and computational modeling as well. Faruque and Humbert\(^22\),\(^23\) modeled the longitudinal and lateral stability of fruit flies around hover using frequency based system identification and quasi-steady aerodynamic modeling. Orlowski and Girard\(^24\) also used a quasi-steady model, and considered the mass of the wings in the equations of motion. They found that treating the mass of the wings as inconsequential could lead to erroneous results. They also demonstrated that full-cycle averaging of the aerodynamic forces can overly simplify the analysis, and they showed that quarter-cycle averaging can produce much more reliable results, with only a small computational cost\(^25\). In a similar vein, Taha et al.\(^26\),\(^27\) have produced several studies that analyze the averaging techniques that are commonly used FWMAV flight dynamics studies. In short, they show that direct averaging of the states and forces of relatively low frequency flappers (e.g. a hawkmoth) can lead to the potentially false
conclusion that such a system is unstable at a fixed point, whereas Floquet theory and their high order averaging technique demonstrates that the system is stable, which agrees with the simulation of the full nonlinear flight dynamics\textsuperscript{27}. They investigate other insects as well, and conclude that as the ratio of flapping frequency to body natural frequency ratio exceeds 100, direct averaging techniques are able to predict if a system will be stable at its fixed points. However, for a bumblebee, they show that direct averaging yields eigenvalues for the unstable mode that are 15\% more than their higher order method predicts. In another work, Taha et al.\textsuperscript{26} demonstrate the stability implications of physical parameters such as the location of the wing root with respect to the body center-of-gravity (CG), and they show how changing flapping frequency and/or mean angle of attack affect each of the stability derivatives in different ways. These studies highlighted the sensitivity of the analysis to both physical and simulation parameters. As such, we carefully construct our model using the wealth of data on the bumblebee in the literature. Additionally, we omit any averaging of the periodic forcing, and couple the aerodynamic, flight dynamics, and when applicable, structural dynamics at each time step. Enforcing this level of consistency is essential to ensuring that other factors do not veil the effect of wing flexibility on the stability of the system.

Sun and Xiong\textsuperscript{28,29} as well as Wu, Zhang and Sun\textsuperscript{30–32} have produced a number of studies that include high fidelity modeling of the Navier-Stokes equations in three dimensions. Their work has examined the hovering and forward flight dynamics of multiple insects of interest to the research community. Highlights pertinent to this study include Sun and Xiong’s work\textsuperscript{28} on small perturbations about equilibrium in the longitudinal plane in hovering bumblebees. Recognizing that even in hover, insects oscillate about an equilibrium point, Wu, Zhang, and Sun\textsuperscript{30} demonstrated the effects of those oscillations on the aerodynamic forces generated. They found that slightly higher angles of attack are required in the presence of such body motion, which translates into slightly higher power demands than previous analysis indicated. Furthermore, Wu and Sun\textsuperscript{32} utilized Floquet stability analysis to study the stability of insects about a periodic equilibrium, concluding that insects with larger wings develop body oscillations large enough to preclude the use of fixed-point stability and control techniques. This finding has significant implications in the field of developing FWMAVs with a proper wing-body mass ratio. Larger wing to body mass ratios appear to be more advantageous from the standpoint of offering the potential to glide and conserve energy as in butterfly flight. However, larger wing mass introduces significantly more complicated flight path considerations and requires the adoption of nonlinear control methods. Liang and Sun\textsuperscript{33} have also demonstrated the ability to couple the full nonlinear equations of motion to a Navier-Stokes solver, and confirmed that insect flight in hover is inherently unstable, and that the pitching moment is the primary cause for the longitudinal instability.

Several researchers have worked on developing control strategies for FWMAVs as well. A first step in adding an element of control is to add a control force term to the linearized equations of motion, which was performed by several of the researchers mentioned previously: Sun and Wang\textsuperscript{20}, Cheng and Deng\textsuperscript{20}, Faruque and Humbert\textsuperscript{23}, Zhang and Sun\textsuperscript{34}, and Liu et al.\textsuperscript{35}. The general theme of these studies is that stabilization control about equilibrium is possible by manipulation of parameters that are commonly observed to be manipulated by insects: stroke amplitude, differential angle of attack, mean stroke angle, and stroke plane orientation. They further observed that these control modes are largely decoupled\textsuperscript{36}. One example from Zhang and Sun\textsuperscript{34} shows that changing stroke amplitude or collective pitch has a significant effect on vertical force, but very small effects on horizontal or lateral force. However flight control that permits maneuvering flight or even simply accomplishing takeoff, mission tasks, and a landing is another matter entirely. By definition, such a high level of control must incorporate closed loop functionality that includes motion sensors and a robust set of control laws. Although several researchers have studied the many aspects of this problem and applied various strategies to it, this remains an open area of research for both biologists and engineers.

In the many studies on the stability and control of insects and FWMAVs in recent years, the majority of the analysis has been based on three significant simplifying assumptions:

1. The inertial effects of wing motion can be safely omitted from the dynamic analysis of the body’s aggregate motion. Although justification of this assumption has been provided\textsuperscript{20}, this assumption has been made primarily in order to allow traditional tools of linear stability analysis to be applied.

2. The aerodynamic forces that are developed across a cycle of wing motion can be averaged and represented by a single, averaged force, which has been justified in several studies\textsuperscript{30,31,36}. When both of these assumptions are made, the equations of motion can be reduced to standard aircraft equations of motion and the host of analysis tools that have been developed for these problems can be applied.

3. The wings of the FWMAV are rigid\textsuperscript{22,24,59,30,33}. This assumption has been made primarily to make the analysis of the system tractable. Whether using a quasi-steady theory or using a Navier-Stokes solver, the studies thus far have assumed that the wings do not deform in the presence of the aerodynamic forces, or that these deformations do not significantly affect the stability of the vehicle.
The objective of this study is to address the third assumption and determine the effect of wing flexibility in the chordwise direction on the longitudinal stability of a hovering bumblebee. Based on Combes and Daniel’s\textsuperscript{37,38} observation that insect wings are orders of magnitudes more flexible in the chordwise direction than in the spanwise direction, we consider only the chordwise flexibility. Prior research indicates that the longitudinal and lateral-directional modes of flapping wing hovering flight are largely decoupled\textsuperscript{16}. We focus on the longitudinal stability of the FWMAV because we expect that considering chordwise flexibility and the resulting passive pitch will have the most significant effect on the pitching moment of the wing. The pitching moment of the wing affects the longitudinal motion of the flapping wing flyer and will likely influence the size and timing of required stabilization control inputs. The following models for the aerodynamic forces and moments on the wing will be compared:

a. A quasi-steady model provided by Sane and Dickinson (for rigid wings) and extended in this study to include pitching moment contributions;

b. A Navier-Stokes equation solution of the rigid wing motion, motivated by Xiong and Sun study of the bumblebee\textsuperscript{28} and Liang and Sun’s coupling of the equations of motion to the Navier-Stokes solver\textsuperscript{33};

c. A fully coupled Navier-Stokes equation and structural dynamics solution of a flexible wing section, allowing for fluid-structure interaction\textsuperscript{13}.

Since this study focuses on longitudinal motion, we assume left-right symmetry and only consider a single wing. The governing equations of motion are developed that include wing inertia, although the simulations presented in this study will not consider the contribution of wing mass on the behavior of the main body. The aerodynamic forces are modeled using a two-dimensional wing section located at 55% span based on the work of Dudley and Ellington\textsuperscript{39}. Both quasi-steady solutions and Navier-Stokes solutions to the aerodynamics are considered. The flexible wing studies are conducted using a fluid-structure solver from Kang et al.\textsuperscript{13} that finds the fully coupled solution of the aerodynamic, elastic, and inertial influences on wing shape, as well as the resulting forces.

II. Methodology

A. Mathematical Model

The equations of motion for a FWMAV are derived from Newton’s second law, accounting for aerodynamic and gravitational forces as well as the inertia of the body and wings and the wing deformation due to the interaction of fluid, inertia, and structural stiffness of the wings. A study of the longitudinal stability of a flapping wing MAV requires three DOFs for the body. The wing motion is prescribed using bio-inspired kinematics that include a flapping angle and a pitching angle (Section II.C). Aerodynamic models include a quasi-steady model and using a well-validated Navier-Stokes solver.

1. Reference frames and key locations

Most studies utilize three main reference frames for the study of flapping wing flight dynamics\textsuperscript{17}: the inertial frame, the body frame and the wing frame. The inertial frame is attached to the surface of the (flat) earth with $x$-$y$-$z$ associated with north-east-down orientations. The body frame is located at the FWMAV center of gravity and is arrived at via 3-2-1 Euler Rotations from the inertial reference frame with the $x_b$ axis aligned longitudinally in the manner depicted in Ellington\textsuperscript{40}, $y_b$ extends laterally to the right, and the $z_b$ axis points down per Fig. 1(a). The rotations between inertial and body frames are very common in the analysis of airplanes and helicopters and, therefore, we refer to Etkin\textsuperscript{41} for the details.

Motivated by the stabilization control studies by Faruque and Humbert\textsuperscript{22} as well as Liu et al.\textsuperscript{35}, we will also assess the effect of orienting the stroke plane angle, which differs from body frame by angle around $y_b$ and $y_{usp}$. The wing frame for the right wing ($w,r$) differs from the upright stroke plane by the following sequence of rotations, depicted in Fig. 1(b):

1. flap angle, $\zeta$, rotating about $y_{usp}$. This is also denoted the $usp^\prime$ frame.
2. vertical deviation angle, $\delta$, rotating about $z_{usp}$.
3. pitch angle, $\alpha$, rotating about $x_{usp}$.

where the subscript “$usp$” stands for upright stroke plane. The relationship between the upright stroke plane and the wing reference frame is given in Fig. 1b. In the current study, the vertical deviation angle is neglected. Furthermore, the motion of the wing is required in two different frames, depending on the aerodynamic model under consideration. The quasi-steady model requires the motion to be expressed in the wing frame. The angle of attack is determined by Eq. (1), where $\vec{v}_w$ is the velocity vector of the quarter chord of the airfoil and includes contributions from the body motion, expressed in the wing frame, and where $f_w$ is the unit vector aligned with the chord of the

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The Navier-Stokes solver requires the wing motion to be expressed in the usp frame, which we also refer to as the cfd frame.

\[
\text{AoA} = \tan^{-1} \left( \frac{\| \mathbf{\dot{v}}_w \times \mathbf{j}_w \|}{\mathbf{\dot{v}}_w \cdot \mathbf{j}_w} \right) 
\]  

(1)

Several reference vectors are required to define the location of key coordinates. The location of the wing root with respect to the body CG is described by \( \mathbf{r}_{w\text{r}} \) (see Fig. 1a), which is expressed in the body frame. Additionally, the location of the wing center of gravity is described by \( \mathbf{r}_{w\text{c}} \) in the wing reference frame. We assume the CG is located at 35% span from the wing root and 25% chord from the leading edge in accordance with values reported by Ennos.\(^4\) Additionally, the wing pitch or torsional axis is allowed to vary. As stated earlier, due to the previous studies that have published important physical, kinematic, and dynamic response of the bumblebees, we also selected the bumblebee for our analysis. Since Sun and Xiong use 30% chord, we conduct analysis using this pitch axis as well. Cases that consider wing flexibility utilize an imposed plunge motion at the leading-edge as well as the resulting passive deformation.

The Reynolds number for this study is based on the wing’s maximum velocity, chord length \( c \), and the kinematic viscosity of air \( \nu \). The maximum velocity of the wing’s aerodynamic center is \( \text{U} = 2\pi f r Z \) and the Reynolds number is \( \text{Re} = \text{Uc}/\nu \). For the current study, the Reynolds number is set to \( 1 \times 10^3 \) for all cases. We nondimensionalize forces and moments by the standard convention of \( C_t = L/(0.5 \rho U^2 S) \) and \( C_m = M/(0.5 \rho U^2 Sc) \) where \( \rho \) is the fluid density and \( S \) is the planform area of a single wing, \( S = Rc \).

Details of the aerodynamic model are provided in Section II.D. The wing’s quarter chord is taken to be the point where the resultant aerodynamic forces act. Weis-Fogh and Lua, Lim, and Yeo determined that the lift force on the wing is proportional to the second moment of wing area, such that the radius to the second moment of wing area serves as the spanwise location of interest for this study. Wings with chordwise flexibility will exhibit changes in camber, which cause the center of pressure to move. Therefore, in studies using solutions to the NS equations, the location of the center of pressure and the aerodynamic moment are calculated directly. Other parameters of interest are listed in Table 1.

2. **Body and wing kinematics**

We assume that the body is rigid. The motion of a rigid body is described by tracking the position, velocity and acceleration of the body’s center of gravity. It is convenient to track the body’s motion in the body-fixed coordinate system. Thus, the kinematics must account for the rotation of the body as well. The velocity and acceleration of the body CG is given as

\[
\mathbf{\dot{v}}_{\text{cg}} = \begin{bmatrix} u & v & w \end{bmatrix}^T
\]  

(2)
\[ \ddot{q}_{\text{cg}} = \frac{d}{dt} (R_{\text{cg}} \dot{v}_{\text{cg}}) = \dot{\dot{v}}_{\text{cg}} + \omega_{\text{cg}} \times \dot{v}_{\text{cg}} \]  

(3)

The kinematics of the wing involve three points of interest: the wing root, the quarter chord, and the wing center of gravity. Referring to Fig. 1(b), we can express the velocity vector of a point \( P \) on the wing as follows (where \( P \) can be substituted for the wing CG, \( w_{\text{cg}} \), or the wing quarter chord, \( w_{\text{qc}} \), as needed)

\[ \ddot{v}_{\text{p}/I} = \ddot{v}_{\text{w}/I} + \ddot{\omega}_{\text{w}/I} \times \ddot{r}_{\text{p}/I} \]

(4)

\[ w_{\text{p}/I} = R_{\text{w} \rightarrow \text{p}} \left( \ddot{v}_{\text{w}/I} + \ddot{\omega}_{\text{w}/I} \times \ddot{r}_{\text{w}/I} \right) + \ddot{r}_{\text{p}/I} + \ddot{\omega}_{\text{w}/I} \times \ddot{r}_{\text{w}/I} \]

(5)

where the \( \ddot{r}_{\text{w}/I} \) term is non-zero only in the case of wing deformation. The acceleration of any point is obtained by taking the time derivative with respect to the inertial reference frame, and applying the transport theorem as necessary. An example is provided in Appendix A.

**Table 1. Morphological parameters used in the current study**

<table>
<thead>
<tr>
<th>symbol</th>
<th>description</th>
<th>value</th>
<th>units</th>
<th>symbol</th>
<th>description</th>
<th>value</th>
<th>units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_b )</td>
<td>mass of body</td>
<td>175</td>
<td>mg</td>
<td>( I_{yy,bg} )</td>
<td>body moment of inertia (pitch)</td>
<td>2.13x10^{-9}</td>
<td>kg-m²</td>
</tr>
<tr>
<td>( L_b )</td>
<td>body length</td>
<td>18.61</td>
<td>mm</td>
<td>( L_{w}/L_b )</td>
<td>distance between CG &amp; wing root</td>
<td>0.21</td>
<td></td>
</tr>
<tr>
<td>( R )</td>
<td>wing length</td>
<td>13.2</td>
<td>mm</td>
<td>( \tilde{r}_2(S) )</td>
<td>% span to 2nd moment of area</td>
<td>0.55</td>
<td></td>
</tr>
<tr>
<td>( c )</td>
<td>mean chord length</td>
<td>4.0</td>
<td>mm</td>
<td>( \tilde{r}_1(m) )</td>
<td>% span to center of mass</td>
<td>0.35</td>
<td></td>
</tr>
<tr>
<td>( S )</td>
<td>area of single wing</td>
<td>53</td>
<td>mm²</td>
<td>( \tilde{c}_1(m) )</td>
<td>% chord to center of mass</td>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>( \chi )</td>
<td>body angle</td>
<td>46.8</td>
<td>°</td>
<td>( \zeta_0 )</td>
<td>mean stroke flapping angle</td>
<td>1 °</td>
<td></td>
</tr>
<tr>
<td>( Re )</td>
<td>Reynolds number</td>
<td>1000</td>
<td></td>
<td>( \beta )</td>
<td>stroke plane inclination</td>
<td>6 °</td>
<td></td>
</tr>
<tr>
<td>( a_d )</td>
<td>down stroke pitch</td>
<td>63</td>
<td>°</td>
<td>( U )</td>
<td>max velocity of ( r_z(S) )</td>
<td>7.15 m/s</td>
<td></td>
</tr>
<tr>
<td>( a_u )</td>
<td>upstroke pitch</td>
<td>-69</td>
<td>°</td>
<td>( f/f_1 )</td>
<td>frequency ratio</td>
<td>0.4</td>
<td></td>
</tr>
<tr>
<td>( Z )</td>
<td>stroke amplitude</td>
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<td>°</td>
<td>( k )</td>
<td>reduced frequency</td>
<td>0.125</td>
<td></td>
</tr>
<tr>
<td>( f )</td>
<td>stroke frequency</td>
<td>155</td>
<td>Hz</td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

**B. Kinetic Analysis**

1. Wing forces and moments

The forces acting on a differential element of the wing, \( dm \), and represented by point \( P \) in Fig. 2(a) are the aerodynamic force and the gravitational force. Additionally, there is a reaction force and moment at the wing root, which can be expressed as

\[ \sum F_{\text{wing}} = \vec{F}_{\text{aero}} + \vec{F}_{\text{grav}} + \vec{F}_{\text{root}} \]  

(6)

The aerodynamic force calculations are described in Section II.D. Their integrated effect is a resultant aerodynamic force that acts at the center of pressure of the wing. Rather than track the center of pressure, however, we calculate the moment about the pitch axis of the wing for convenience. From Newton’s Third Law, the force of the body on the wing, \( w \vec{F}_{\text{root}} \), is equal and opposite to the force of the wing on the body, \( w \vec{F}_{\text{root}} \). Thus, the net force of the wing on the body is equal to the aerodynamic and gravitational forces minus the inertia that arises from the wing’s motion per Eq. (7).

\[ w \vec{F}_{\text{root}} = w \vec{F}_{\text{aero}} + m \ddot{g} - m \left( \ddot{\vec{a}}_{\text{wg}/a} + 2 \ddot{\omega}_{w/I} \times \vec{v}_{\text{w}/a} + \ddot{\vec{v}}_{\text{w}/a} + \ddot{\omega}_{w/I} \times \ddot{\vec{r}}_{w/I} + \ddot{\vec{r}}_{w/I} \right) \]

\[ -m R_{\text{h} \rightarrow \text{w}} \left( \ddot{\vec{a}}_{\text{cg}} + \ddot{\omega}_{b/I} \times \ddot{\vec{v}}_{\text{cg}} + \ddot{\vec{v}}_{\text{cg}} + \ddot{\omega}_{b/I} \times \ddot{\vec{r}}_{b/I} + \ddot{\vec{r}}_{b/I} \right) \]

(7)
Equation (8) details the various external moments that combine to change the angular momentum of the wing.

$$\sum \omega \cdot \mathbf{M}_a = u \dot{\mathbf{M}}_{\text{aero}} + u \dot{\mathbf{M}}_{\text{Rot}} + \dot{\mathbf{r}}_{\text{avg}/o} \times \mathbf{w} + u \dot{\mathbf{r}}_{\text{avg}/o} \times \dot{\mathbf{F}}$$  \hspace{1cm} (8)

Therefore, the moment of the wing on the body consists of the aerodynamic and gravitational moment minus the changes in angular momentum of the wing, which is given in Eq. (9).

$$w \dot{\mathbf{M}}_{\text{Rot}} = w \dot{\mathbf{M}}_{\text{aero}} + \dot{\mathbf{r}}_{\text{avg}/o} \times \dot{\mathbf{F}}_{\text{aero}} + \dot{\mathbf{r}}_{\text{avg}/o} \times \mathbf{w} - \left( w \mathbf{L}_{\text{avg}} \hat{\omega}_w + \dot{\mathbf{r}}_{\text{avg}} \times \mathbf{w} \right)$$  \hspace{1cm} (9)

Figure 2. a) Forces, moments and geometric parameters for the bumblebee wing; b) Forces, moments and geometric parameters for the body.

2. Body forces and moments

The weight of the body acts through the body CG, and since the moments are taken about the CG, the weight of the body does not contribute to the moment. Since the motion of the body in hover is considered small relative to the velocities developed on the wing, we neglect the aerodynamic forces and moments developed on the body. The equations of motion are obtained by substituting the appropriate terms that have been derived above into Eqs. (10) and (11). This process is detailed in Appendix A.

$$\sum \mathbf{F}_{\text{body}} = \dot{\mathbf{F}}_{\text{grav}} + \dot{\mathbf{F}}_{\text{rot}} = m_b \dot{\mathbf{G}}_{\text{eg}}$$  \hspace{1cm} (10)

$$\sum \dot{\mathbf{M}}_{\text{eg}} = \dot{\mathbf{M}}_{\text{Rot}} + \dot{\mathbf{r}}_{\text{avg}/o} \times \dot{\mathbf{F}}_{\text{aero}}_{\text{avg}} = \mathbf{L}_{\text{b,eg}} \hat{\omega}_b + \dot{\mathbf{r}}_{\text{avg}} \times \mathbf{L}_{\text{b,eg}} \hat{\omega}_b$$  \hspace{1cm} (11)

3. Solving the equations of motion

The flight dynamic equations of motion of the body are nonlinear equations. All of the highest derivative terms, however, are linear, and the coupling of the highest derivative terms is through an equivalent mass matrix, $\mathbf{H}$ which is positive definite. Further, when the inertia of the wing is neglected and the inertia tensors do not have products of inertia, this becomes a diagonal matrix. In the current study, we neglect wing mass in the solutions of the equations of motion based on the findings of Sun, Wang, and Zhang. In comparing the response of the system to rigid wings against flexible wings, we do not consider wing mass so that the effect of adding wing mass does not mask the effect of flexibility (wing mass is included in calculating the flexible wing’s response, but not the body’s response). However, in future studies the inertia contribution of the wings will be assessed.

This system can be represented via a state vector, which is defined in Eq. (12) where $u$, $v$, and $w$ represent the velocity components of the body, $p$, $q$ and $r$ represent the rotation rates of the body about $x_b$, $y_b$, and $z_b$, $x_{cg}$, $y_{cg}$, and $z_{cg}$ represent the displacement of the body in the inertial frame, and $\phi$, $\theta$, and $\psi$ represent the orientation of the body. The control inputs to the system are based on the selection used by Sun and Xiong and are defined in Eq. (13) where $\alpha_d$, $\alpha_u$, and $\zeta$ are the down-stroke pitch angle, upstroke pitch angle, and flapping angle at the mid-stroke, respectively.
\[
\dot{x} = \begin{bmatrix} u & v & w & p & q & r & x_{eg} & y_{eg} & z_{eg} & \phi & \theta & \psi \end{bmatrix}^T \tag{12}
\]

\[
\dot{\mathbf{u}}(t) = \begin{bmatrix} \alpha_d & \alpha_u & \zeta \end{bmatrix}^T \tag{13}
\]

The solution to these equations is obtained by isolating the derivatives of the state vector, which results in a coupled system of nonlinear equations. These equations are solved simultaneously in order to determine the acceleration of the body. Once the acceleration of the body is known, its response to perturbations in each axis of interest can be determined and analyzed, and it can be integrated numerically in time to determine the longitudinal response of the system. A more complete description of the terms included in Eq. (14) is provided in Appendix A.

\[
\hat{x} = H^{-1} f(\bar{x}, \bar{u}) \tag{14}
\]

C. Prescribed Wing Motion

The wing motion is prescribed with respect to the body. Three popular wing kinematics models exist: i) observed kinematics in nature, ii) simple harmonic functions, and iii) special functions suited for control purposes\(^{17}\). We chose to model the kinematics to match the bio-inspired flapping and pitching presented in Sane and Dickinson\(^2\) and used extensively by Sun and co-workers\(^{28,30,46}\).

1. Flapping motion

Equation (15) describes the flapping motion of the wing and was adapted from Sun et al.\(^{29}\). \(Z\) is the amplitude (half peak-to-peak) of the flapping motion, \(f\) is the frequency of the motion, and \(\zeta\) is the mean flapping angle in the upright stroke plane. The time rates of flapping angle are readily obtained by taking the appropriate time derivatives. In each of these kinematic equations, time is nondimensionalized with respect to the full wing-beat period using \(\tau = ft\). Since the timing of the pitching motion is defined relative to the flapping motion, there is no need to specify a phase delay for the flapping motion.

\[
\zeta(\tau) = Z \cos 2\pi \tau + \zeta \tag{15}
\]

2. Pitching motion

In addition to the flapping motion, the kinematical description of the pitching motion about the wing’s spanwise axis of rotation is important for both force generation and control. In general pitching can be passive or active. In flexible wing studies, we do not consider active pitching. This yields a deformation that can be described in terms of a passive rotation. For actively rotated wings, however, the timing and during of the rotation relative to the flapping motion has a significant impact of the magnitude and direction of the aerodynamic forces. In this study, rigid wing analyses will only consider symmetric pitching motion, unless specifically mentioned to the contrary. The pitching relations repeated in Eq. (16) are based on those given by Liang and Sun\(^{33}\) and used by several other studies by Sun and coworkers, where \(A\) is the amplitude (half peak-to-peak), \(\tau\) is the time nondimensionalized by the wing beat period, and \(\Delta \tau\) is the duration of wing rotation (which are also used in Sun and Xiong\(^{28}\)). Using the same wing kinematics as these previous studies facilitates comparison between the studies.

\[
\alpha(\tau) = \alpha_d + \frac{A}{\Delta \tau} \begin{cases} (\tau - \tau_1) - \frac{\Delta \tau}{2\pi} \sin \left( \frac{2\pi}{\Delta \tau} (\tau - \tau_1) \right) & \tau \leq \tau_1 \\ (\tau - \tau_2) - \frac{\Delta \tau}{2\pi} \sin \left( \frac{2\pi}{\Delta \tau} (\tau - \tau_2) \right) & \tau_2 \leq \tau \leq \tau_2 + \Delta \tau_2 \\ (\tau - \tau_3) - \frac{\Delta \tau}{2\pi} \sin \left( \frac{2\pi}{\Delta \tau} (\tau - \tau_3) \right) & \tau_3 \geq \tau \end{cases} \tag{16}
\]

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D. Aerodynamic Model

The forces and moments generated by the flow of the air over the wings and body are the most significant source of forces in this problem, so care must be taken to model their effects accurately. Results are presented based on three different models for the aerodynamic forces. These models are tied to the assumptions concerning the properties of the wing, in particular, whether the wing is rigid or flexible.

a. A quasi-steady model provided by Sane and Dickinson\(^4\) (for rigid wings)
b. A CFD solution of the rigid wing motion, solving for the Navier-Stokes equations.
c. A CFD-CSD solution of a flexible wing, solving for the Navier-Stokes equations and linear beam equations in a tightly coupled manner.

1. Quasi-steady aerodynamic model

The Sane and Dickinson model has been used extensively in the past by others, and it will be incorporated into this study as well. Their model\(^4\) attempts to capture the quasi-steady contributions of translational lift, \(F_{\text{trans}}\), rotational lift from circulation, \(F_{\text{rot}}\) and added mass force, \(F_a\) per Eq. (17). Although Sane and Dickinson\(^4\) identify that additional forces are generated via wake capture, this highly nonlinear phenomena is not amenable to an algebraic representation, and it is omitted from the model. The translational lift and drag coefficients are provided by fitting an expression to experimental results for a range of angles of attack: \(\text{AoA} = -10^\circ \text{ to } 90^\circ\), resulting in Eqs. (18) and (19).

\[
F_{\text{w,aero}} = F_a + F_{\text{trans}} + F_{\text{rot}} \tag{17}
\]

\[
\vec{F}_{\text{trans}} = \begin{bmatrix} L_{\text{trans}} \\ D_{\text{trans}} \end{bmatrix} = \frac{1}{2} \rho c_w^2 R c_L \begin{bmatrix} c_L \\ c_D \end{bmatrix} \tag{18}
\]

\[
c_L = 0.225 + 1.58\sin(2.13\text{AoA} - 7.2) \tag{19}
\]

\[
c_D = 1.92 - 1.55\cos(2.04\text{AoA} - 9.82) \tag{19}
\]

The added mass term is given by Eq. (20), and the terms in Eq. (21) are defined by Ellington\(^40\) and repeated here for convenience. In these equations, \(\hat{c}\) is the local chord and \(\hat{r}\) is the spanwise location, both nondimensionalized by the span of a single wing.

\[
F_a = \pi \rho R c^3 \left[ \frac{R}{4} \left( \hat{c} \sin \text{AoA} + \hat{\zeta} \hat{\alpha} \cos \text{AoA} \right) \nu \hat{r}_i(v) - \frac{\hat{\alpha}}{16} \right] \tag{20}
\]

\[
\nu \hat{r}_i(v) = \int_0^\hat{r} \hat{c}^2 \hat{r} d\hat{r} \quad \text{and} \quad \hat{v} = \int_0^\hat{r} \hat{c}^2 \hat{v} d\hat{r} \tag{21}
\]

The aerodynamic moment about any point can be determined as long as the point of application for each aerodynamic force is known. The circulatory lift terms are applied at the quarter chord, and there is a residual pitching moment that arises due to wing rotation per Eq. (22), given by Leishman\(^47\).

\[
\vec{M}_{\text{w,circ}} = -\left[ \frac{\pi}{16} \rho c^3 r_i R \phi \hat{\alpha} \quad 0 \quad 0 \right]^T \tag{22}
\]

The increment in lift due to added mass is taken to act at the mid-chord so the moments are given by Eq. (23). Additionally, Bisplinghoff\(^48\) shows that pure plunging motion develops a pressure distribution around the airfoil that produces no net lift but produces a destabilizing moment similar to that experienced by a 3-D body in pure translation. Fung\(^49\) also demonstrates that a nose down couple is generated by \(\hat{\alpha}\). These terms are added to Eq. (22) to yield the pure couple that the wing experiences. The moments that arise due to \(\vec{F}_{\text{w,aero}}\) are tracked as separate terms in the rotational equations of motion.

\[
\vec{M}_{\text{w,aero}} = 0.25 c_f \dot{\omega} \times \vec{F}_{\text{w,aero}} + R \left( \frac{\pi \rho c^2}{4} U \dot{h} - \frac{\pi \rho h^2}{8} \hat{\alpha} \right) \dot{\omega} \tag{23}
\]
2. High fidelity model for rigid wing

In order to provide a point of comparison against the coupled CFD-CSD solver that will be discussed below, the rigid wing is also modeled using a well-validated Navier-Stokes solver. The case setup requires the kinematics of the wing as described in Section II.C to be imposed on a quiescent fluid. The fluid response and resulting viscous stress and pressure distributions on the body are described by the unsteady, incompressible Navier-Stokes equations given in Eq. (24)

\[ \nabla^* \cdot \vec{u}^* = 0 \\
\frac{k}{\pi} \frac{\partial \vec{u}^*}{\partial t} + (\vec{u}^* \cdot \nabla^*) \vec{u}^* = -\nabla^* \rho^* + \frac{1}{Re} \Delta \vec{u}^* \]

where the asterisk (*) indicates variables that have been nondimensionalized with the reference velocity, \( U \), 1/f, and \( c \) (all defined in Section II.C.4) at a Reynolds number of \( 1 \times 10^3 \), which is required for similitude at the bumblebee scale. The reduced frequency, \( k \), in hover reduces to a geometric relationship that is governed by the stroke amplitude: \( k = \pi f c / U = \pi c / (2 Z_{r2}) \). The wing thickness-chord ratio, \( h_{*} \), is taken to be \( 1.5 \times 10^{-3} \) based on the observations of Lehman et al.\(^{50}\) and Lehman and Dickinson.\(^{51}\) These equations are solved in two dimensions using a structured, finite-volume, pressure-based Navier-Stokes equation solver used extensively in flapping wing studies by Shyy and coworkers.\(^{52,53}\) The pressure and shear stress distributions are integrated to yield the forces and moments of the wing about the wing’s pitch axis at each time step. Any use of the term CFD in this paper refers to this model.

3. Coupled CFD-CSD model

The case setup for the coupled Navier-Stokes and Euler-Bernoulli elastic beam solver follows the work of Kang et al.\(^{13}\), Kang and Shyy,\(^{15}\), and Sridhar and Kang.\(^{10}\) and is summarized briefly here. With the addition of wing flexibility, the density and stiffness of the wing become important. The density ratio, \( \rho^* = \rho_w / \rho_f \), is taken to be \( 1 \times 10^{-1} \), where \( \rho_w \) is the wing density and \( \rho_f \) is the air density. The stiffness is defined by the frequency ratio, \( f/f_i \), which is set to 0.4 based on Sridhar and Kang,\(^{10}\) who found that this frequency ratio offers near-optimal lift production. Since chordwise flexibility is of interest, the complicated 3D anisotropic topology of an actual insect wing is generalized by the two dimensional bending of an insect wing, a process modeled via the Euler-Bernoulli beam equation given in Eq. (25)

\[ \Pi_o \frac{\partial^2 v^*}{\partial t^2} + \Pi_i \Delta^* v^* = F^* \]

where \( v \) is the transverse displacement due to bending. \( \Pi_o = \rho^* h_{*}^* (k/\pi)^3 \) is the inertia of the wing normalized by the fluid dynamic variables, \( \Pi_i = Eh_{*} / (12 \rho_f U^2) \) is the wing stiffness normalized by the fluid dynamic variables, and \( F^* \) is the distributed normal fluid force per unit length on the wing, given by \( F^* = F / (\rho_f U^2)^{13} \). We use a structured, pressure-based finite volume solver to solve Eq. (24) that governs the motion of the fluid\(^{52,53}\). We solve Eq. (25) using an in-house finite element representation of an Euler-Bernoulli beam model. Any use of the term CSD in this paper refers to this model. Equations (24) and (25) are solved independently, and coupling is achieved via a time-domain partitioned process. At each time step the fluid and structural solutions are iterated until sufficient convergence is reached. Details of the fluid-structure interaction and careful validation studies against well-documented experimental results are shown in previous studies\(^{13,54-56}\). The computational grid is re-meshed whenever the wing moves or deforms using the radial basis function interpolation scheme.\(^{13,57}\)

4. Coupled CFD-EOM and CFD-CSD-EOM model

The method for finding the solution to the flight dynamics equations was provided in Section II.B.3 and is expanded in Appendix A. The aerodynamic forces and moments are calculated directly by the CFD or CFD-CSD solvers by integrating the pressure and shear forces on the wing. The moments are summed about the pitch axis that was selected during case set-up. They are nondimensionalized and provided to the equations of motion along with the current state vector. The equations of motion utilize dimensionalized forms of the forces based on the insect-specific parameters, and are solved to yield the body accelerations for each time step. The body accelerations are integrated in time using a second-order Adams Bashforth scheme to yield the body velocities and displacements. The displacements with respect to the inertial frame are then transformed back into the local c/f/d frame and provided to the Navier-Stokes solver. The body motion is combined with the change in position and angle due to the
prescribed wing motion at each step time step, and the computational grid is re-meshed at each time step. Thus, the solution is a tight three-way coupling of the governing equations, depicted in Fig. 3. This tight coupling is selected for several reasons. First, Orlowski and Girard\textsuperscript{25} as well as Taha, et al.\textsuperscript{27} describe the various ways that averaging can bias the solution, and we wish to avoid any artifacts that might obscure the effect of wing flexibility on the nature of the solution. Secondly, averaging the forces or accelerations introduced unacceptable oscillations in the simulation, a feature that will be discussed further in Section III.C.

**Figure 3. Schematic of three-way coupled CFD-CSD-EOM solver.**

### III. Results and Discussion

Several lines of inquiry are followed in this paper in the development of multi-fidelity tools needed to analyze the flight mechanics of flapping wing MAVs. Much of this section is devoted to validating our models. We first present the results of the flight dynamics studies that were performed using a quasi-steady model based on Sane and Dickinson as described in Section II.C. We also present validation of the rigid-wing Navier-Stokes solutions as well as a discussion of the grid and time-step sensitivity analysis required at the Reynolds number pertinent to a bumblebee. We then compare the results of the equations of motion with other studies that analyzed high fidelity aerodynamic models with the EOMs. Finally, we compare the fluid-structure interaction model with previously published work, and then demonstrate the effect of the flexibility on the longitudinal response of the system starting at hover. We also compare the quasi-steady model with the rigid and flexible Navier-Stokes solutions.

#### A. Quasi-steady Analysis

As mentioned in Section I, many different studies used a quasi-steady aerodynamic model to predict the aerodynamic forces and moments. Most of these studies either utilize a rigid-body simplification of the equations of motion or average the forces across an entire wingbeat cycle or both. The equations presented in Section II.D.1 represent the fully-coupled multibody dynamic equations of motion. In this study, the flight dynamics and aerodynamics model communicate at each time step. Resultant forces and moments are fed into the fluid dynamics solver. In turn, body angles and rates are added to the circulatory lift, rotational lift, and added mass terms. Figure 4 demonstrates the agreement between our implementation of the model provided by Sane and Dickinson\textsuperscript{4} for symmetric pitching motion.
Figure 4. Components of (a) lift and (b) drag. Solid lines depict the quasi-steady model from Sane and Dickinson\textsuperscript{4} used in the current study. Dashed lines represent the quasi-steady model from Sane and Dickinson\textsuperscript{4}. Blue lines (―) are translational forces, brown lines (―) are added mass forces, and black lines (―) are the rotational forces.

The primary benefit of using a quasi-steady model is the speed with which analysis can be performed. As a result, the use of the quasi-steady model allowed us to find an equilibrium position rapidly. Thus, after confirming the validity of our quasi-steady aerodynamic model, we assess the ability of our model to solve the longitudinal flight dynamic equations of motion in two ways. The first is by analyzing the longitudinal response of the nonlinear flight dynamics model to various initial conditions, and the second is by analyzing the linearized system itself near an equilibrium point.

The equilibrium position is solved via a multi-DOF Newton-Raphson method\textsuperscript{58}. The system is linearized by introducing perturbations to both the states and controls, to obtain the Jacobian of the system at that point. We used central difference method to determine each term, and we determined that a disturbance of 1\% gave the best convergence behavior. This process is iterated until the two-norm of the cycle-averaged acceleration vector, \( \dot{\mathbf{x}} \), < \( 1 \times 10^{-2} \), indicating an equilibrium point of the linearized system.

Once the system is near equilibrium, the linearized system, represented by Eq. (26) can be used to conduct further analysis on the open loop stability of the system. In this arrangement, the system matrix, \( \mathbf{A} \), describes the rate response of each DOF to a disturbance in each of the state variables in \( \mathbf{x} \). When the system is near equilibrium, this matrix contains the stability derivatives of the aircraft, where \( X_{\alpha} = \partial F_{\alpha} / \partial \alpha \), \( X_{w} = \partial F_{w} / \partial w \), etc. Because only longitudinal motion is allowed, only four states affect the rate vector per Eq. (27), which is the same result used by other researchers investigating longitudinal stability, particularly Sun and Xiong\textsuperscript{28} and similar to that used by Faruque and Humbert\textsuperscript{22}.

\[
\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{26}
\]

\[
\mathbf{A} = \begin{bmatrix}
\frac{1}{m} X_{\alpha} & \frac{1}{m} X_{w} & \frac{1}{m} X_{q} & -g \\
\frac{1}{m} Z_{\alpha} & \frac{1}{m} Z_{w} & \frac{1}{m} Z_{q} & 0 \\
\frac{1}{I_{yy}} M_{\alpha} & \frac{1}{I_{yy}} M_{w} & \frac{1}{I_{yy}} M_{q} & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} \tag{27}
\]

The eigenvalues and eigenvectors of the system matrix describe the behavior of the natural modes of the system near equilibrium to an external disturbance. In Table 2, we compare the eigenvalues of our system to those published by Xiong and Sun\textsuperscript{28}. Additionally, the elements of the system’s eigenvectors are listed in Table 2 by their
degree of freedom (e.g. $\delta u^+$, $\delta w^+$, etc.). The same insect parameters and kinematics are used. Additionally, the eigenvalues are plotted on the imaginary plane in Figure 5. Sun and Xiong\textsuperscript{28} identified three modes of the system, which they labeled a fast subsidence mode, a slow subsidence mode, and an unstable oscillatory mode. The same three modes were observed using our quasi-steady model with very similar decay/growth rates and mode shapes, and a similar frequency in the oscillatory mode. The values are very similar despite the use of a Navier-Stokes equation solver in Sun and Xiong\textsuperscript{28}.

Table 2. Comparison of the system response to external disturbance for the current quasi-steady results and the Navier-Stokes solutions by Sun and Xiong\textsuperscript{28}.

<table>
<thead>
<tr>
<th></th>
<th>Fast Subsidence</th>
<th>Slow Subsidence</th>
<th>Unstable Oscillatory Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Sun &amp; Xiong\textsuperscript{28}</td>
<td>Current QS</td>
<td>Sun &amp; Xiong\textsuperscript{28}</td>
</tr>
<tr>
<td>eigenvalue</td>
<td>-0.1931</td>
<td>-0.1549</td>
<td>-0.011</td>
</tr>
<tr>
<td>$\delta u^+$</td>
<td>0.0722</td>
<td>0.0843</td>
<td>0.0788</td>
</tr>
<tr>
<td>$\delta w^+$</td>
<td>-0.0002</td>
<td>-0.0062</td>
<td>0.9953</td>
</tr>
<tr>
<td>$\delta q^+$</td>
<td>-0.1891</td>
<td>-0.1525</td>
<td>-6.0x10^{-4}</td>
</tr>
<tr>
<td>$\delta \theta^+$</td>
<td>0.9793</td>
<td>0.9847</td>
<td>0.0557</td>
</tr>
</tbody>
</table>

As has been discussed elsewhere\textsuperscript{18,28}, the eigenvalues of a linear system determine whether the system is stable or unstable. The eigenvalue dictates the growth or decay of a particular mode: a positive eigenvalue indicates growth of a mode and a negative eigenvalue indicates decay. If the imaginary part of a complex conjugate pair of eigenvalues is non-zero, the system’s response will oscillate as it decays or grows. The eigenvectors denote the level of participation of each degree of freedom within a natural mode. For example, we predict the unstable oscillatory mode to have an eigenvalue pair of $0.0468±0.1177i$. Since the real part is greater than zero, the response of this mode will grow as it oscillates. The eigenvectors reveal that the unstable oscillatory mode is dominated by the pitch degree of freedom since it is an order of magnitude larger than the other degrees of freedom. On the other hand, the other two eigenvalues are real and negative, indicating that those modes will experience exponential decay without oscillation.

In addition to the response to external disturbances, it is important to characterize the response of the vehicle to control inputs. Although many different parameters could be chosen as control inputs for the bumblebee including stroke plane angle, timing and duration of wing rotation, flapping amplitude, flapping frequency, vertical deviation angle, etc. we chose to use the upstroke and down stroke pitch angle, $\alpha_u$ and $\alpha_d$, and the mid-stroke flapping angle, $\zeta$. These are the same input parameters as those used by Sun and Xiong\textsuperscript{28} to facilitate comparison and validation of our results. The control derivatives are per Eq. 28.

$$B = \begin{bmatrix} \frac{\partial \bar{u}}{\partial \alpha_d} & \frac{\partial \bar{u}}{\partial \alpha_u} & \frac{\partial \bar{u}}{\partial \bar{q}} \\ \frac{\partial \bar{w}}{\partial \alpha_d} & \frac{\partial \bar{w}}{\partial \alpha_u} & \frac{\partial \bar{w}}{\partial \bar{q}} \\ \frac{\partial \bar{q}}{\partial \alpha_d} & \frac{\partial \bar{q}}{\partial \alpha_u} & \frac{\partial \bar{q}}{\partial \bar{q}} \end{bmatrix}$$

(28)
Trim was established for $\alpha_u = 75.4^\circ$, $\alpha_d = -69.6^\circ$ and $\zeta = 0.38^\circ$. As in Sun and Xiong\textsuperscript{28}, the position of the mean flapping angle has significant control authority over the pitching motion of the insect, and the pitch angles primarily control the vertical and horizontal forces. A comparison of the quasi-steady model vs. Xiong and Sun’s results for a disturbance in the positive $x$-direction is depicted in Fig. 6.

In summary, the quasi-steady portion of this study allowed us to validate the solution of the equations of motion and to learn how the system responds to various control inputs and external disturbances. Of note is the initial input for body pitch rate, $q$, needed to be approximately 3 radians per second, which may seem high, but is equal to less than 2 degrees per period. For the results presented in Fig. 6, the initial states and controls are provided in Table 3. Since the motion about equilibrium is oscillatory in nature, it was not possible to find a cycle-averaged value of acceleration equal to zero without including these initial conditions.

### B. Solutions to the Coupled Navier-Stokes and Flight Dynamics Equations

The Navier-Stokes equation solver has been previously validated at low Reynolds numbers for example in the work of Tang et al.\textsuperscript{52} In order to assess the effects of grid resolution and time step size at the bumblebee scale at $Re = 1 \times 10^3$, a grid convergence study was conducted to determine the optimal grid resolution, clustering scheme, and domain size. This analysis as well as the computational setup are presented in Appendix B.

Since the primary purpose of the study is to analyze the effect of wing flexibility, we need to compare the flexible results with results obtained by high fidelity simulation on a rigid wing. Additionally, we wish to compare the results we obtain for rigid wings to those in the literature. In particular, we utilize the parameters obtained by Sun and Xiong for equilibrium at a hover and also subject to a gust in each degree of freedom in order to assess the stability of the linearized model. Because our solution to the NS equations is based on 2D flow, slightly different results are inevitable compared to Sun and Xiong’s 3D simulations\textsuperscript{28}. The results of a simulation that compares the difference in $C_L$, $C_D$, and $C_M$ between pure hovering and a disturbance in the positive $x_d$ direction are plotted in Fig. 7 where $\Delta C_L = C_{L,u=0.05U} - C_{L,hover}$ and the differences for the other coefficients are defined in the same way. The quasi-steady results are also included to highlight the differences in modeling that result from using a Navier-Stokes solution. The current study shows reasonably good agreement with both the Navier-Stokes simulations performed by Xiong and Sun\textsuperscript{28} and also the quasi-steady result mentioned earlier, particularly in the mid-stroke where the wing pitch takes on values prescribed by Sun and Xiong\textsuperscript{28}. The most noticeable differences arise in the portions of the period where the wing must actuate from $+63^\circ$ in one direction to $-69^\circ$ in the other in 22% of the stroke. This rapid transient maneuver produces significant forces due to added mass, rotational lift and wake-capture mechanisms. Figure 8 depicts the vorticity in the flow at a representative stroke immediately following stroke reversal ($\tau = 0.54$). The vorticity upstream of the wing was shed during the previous stroke, and the wing is moving from left to right. As the wing interacts with these flow structures, the large changes in $\Delta C_L$ that appear in Fig. 7 are produced.
differences between the Navier-Stokes simulations and the quasi-steady model are therefore due to the nonlinear vorticity dynamics in the flow field and any history effects, where both are neglected by the quasi-steady model. The time history of $\Delta C_L$ in Fig. 7 is the difference between the horizontal wind gust and hovering cases in Fig. 9. Figure 9 also depicts the lift histories for the simulations where the system was perturbed in the vertical and rotational degrees of freedom as well. This permits a comparison of the lift that is produced in each case and facilitates an explanation of the differences. The first observation is that in each case, the lift histories follow the same qualitative trend. Considering the disturbance in the positive $x$ direction, the disturbance is modeled with a flow from the head to tail of the body such that $u = 0.05U$. In the first half stroke the wing advances into the flow, increasing the relative wind on the advancing wing and producing more lift. In the second half stroke the wing is retreating with the flow, the relative wind is lower, and lower lift results. During the first wing rotation, the wing rotates into the relative wind, increasing the effect of the rotational and added mass lift mechanisms; during the second rotation, the wing rotates with the flow, and these effects are decreased.

A disturbance in the positive $z$-direction is modeled with an inflow from the bottom to the top of the insect such that $w = 0.05U$. As expected the lift is enhanced throughout the stroke as a result. In the translational portions of the stroke (from approximately $\tau = 0.1-0.4$ and $\tau = 0.6-0.9$), the flow from the bottom increases the effective angle of attack, which increases the lift. In both of the rotational portions of the stroke, the flow from the bottom increases the lift. Although all of the mechanisms are not clear, one reason for this increase in rotational lift is that the vorticities that are shed in earlier strokes are not able to convect downward as readily, and they remain closer to the wing, particularly during the second half of the wing’s rotation, which occurs when $\tau = 0.0-0.1$ and $0.5-0.6$.

A disturbance in the body’s positive rotational direction is modeled by imparting a constant rotation rate of $0.07f = 4^\circ$/period (nose up). Because the wing root is located $0.75c$ above the body center of mass, this rotation of the body imparts a rearward motion of the wing. This has essentially the opposite effect as the gust coming from the front of the insect that was discussed earlier. This is most readily seen in the translational portions of the stroke where the lift is less than the hover case in the first half of the stroke, and more than the hover case in the second half of the stroke due to the difference in relative wind that the wing experiences. However, in the first rotational portion of the stroke, the rotation of the insect adds to the rotation rate of the wing, and the lift during this portion of the stroke is enhanced. In the second rotational phase, the rotation rate of the insect subtracts from the rotation rate of the wing and less rotational lift is produced.

The differences in the forces and moments developed by the wing directly affect the system’s response to the aerodynamic forcing. Table 4 details the eigenvalues of the linearized system in comparison to those reported by Xiong and Sun$^{28}$, and the eigenvalues are plotting on the Re-Im axis as well in Fig. 10. The magnitude of each of the eigenvalues in the current analysis is larger than that predicted by Sun and Xiong$^{28}$ and larger than those predicted by the quasi-steady analysis as well. This is most likely due to the larger difference in aerodynamic forces and moments that are clearly seen in Fig. 9.
Figure 8. Vorticity in the flow field just after stroke reversal at $\tau = 0.54$. The wing is translating to the right and rotating clockwise. The solid white region represents the rigid wing. Vorticity is nondimensionalized by $U/c$.

Figure 9. Lift coefficient history, $C_L$, for hovering and a disturbance in each body direction.
These can arise from the 2D versus 3D modeling differences mentioned previously. We also use a grid with over twice the number cells in the chordwise direction, and we use an outer domain boundary of 50 chord lengths vice 20 in Sun and Xiong.28

In spite of these differences, however, the qualitative response to the disturbances is the same—a fast subsidence mode, a slow subsidence mode, and an unstable oscillatory mode are observed once again. Taha et al.26 report on the importance of the size of physical parameters such as location of the axis of wing rotation, location of the wing root, and other distances that create moments about the CG. Since the same physical parameters were used in our study as those used by Sun and Xiong,28, some of the similarity in the overall system response is likely due to this consistency in the modeling. However, the instability of the oscillatory mode of the linearized system that is dominated by the body pitch degree of freedom is consistent with other studies as well16.

Table 4. Comparison of the system response to external disturbance for the current Navier-Stokes solutions and the Navier-Stokes solutions by Sun and Xiong.

<table>
<thead>
<tr>
<th></th>
<th>Fast Subsidence</th>
<th>Slow Subsidence</th>
<th>Unstable Oscillatory Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun &amp; Xiong</td>
<td>-0.1931</td>
<td>-0.011</td>
<td>0.0447 ± 0.1279i</td>
</tr>
<tr>
<td>Current, NS</td>
<td>-0.2064</td>
<td>-0.0091</td>
<td>0.0388 ± 0.0905i</td>
</tr>
<tr>
<td>δu ^+</td>
<td>0.0722</td>
<td>0.0788</td>
<td>0.9953 ± 0.0002i</td>
</tr>
<tr>
<td>δw ^+</td>
<td>-0.0002</td>
<td>-0.0034</td>
<td>-0.0023 ± 0.0441</td>
</tr>
<tr>
<td>δq ^+</td>
<td>-0.1891</td>
<td>-0.2018</td>
<td>-6.0x10^-4</td>
</tr>
<tr>
<td>δθ ^+</td>
<td>0.9793</td>
<td>0.977</td>
<td>0.2570 ± 0.9861</td>
</tr>
</tbody>
</table>

C. Solutions to the Coupled Navier-Stokes, Linear Beam, and Flight Dynamics Equations

1. Cause and effect of resulting wing motion

The longitudinal response of the insect modeled with rigid wings is provided in Section III.B. In order to analyze the influence of the wing flexibility on the stability of the system near equilibrium, we consider a tightly coupled fluid/structure/flight dynamics interaction model (CFD-CSD-EOM). Validation of the Navier-Stokes equation solver (CFD), fully coupled with a linear beam model (CSD) has been previously conducted. We start the CFD simulation with the wing at rest. In order to allow the effects of the initial transients on both the flow solution and the flexible beam’s response to dissipate, the integration of the equations of motion (EOM) begins at the start of the eighth flapping period. Figure 11 illustrates the lift history predicted by the fully coupled CFD-CSD-EOM. The first two cycles clearly show transient effects from the simulation startup. The magnitude of the lift peaks in cycles 2-8 shows more consistency, but the number, magnitude and phasing of the force peaks do not become approximately periodic until after eight cycles.

The response of the flexible wing during the first half-stroke appears in Fig. 12. The physical underpinnings of this motion have been discussed in our earlier work, but highlights pertinent to this study are summarized briefly here. Figure 12 shows twelve equally timed snapshots of the wing shape and orientation during a forward stroke of the plunge motion. Deformation of flexible wing with passive pitch angle and actively rotated rigid wing are shown in red and black respectively. The red dot indicates the leading edge of both rigid and flexible wings. While chord
length of rigid wing remains fixed during the motion, the linear Euler-Bernoulli beam model only accounts for the transverse wing deformation, leading to an elongated chord length. We define the force coefficients based on the instantaneous chord\textsuperscript{15}. The flapping amplitude is $Z = 127^\circ$, reduced frequency is $k = 0.124$, and the frequency ratio, the ratio between the flapping frequency and the first natural frequency in the chordwise direction is $f/f_1 = 0.4$.

At the beginning of the stroke, both rigid and flexible wings are vertically oriented. During the forward stroke, maximum wing rotation occurring near the mid-stroke. The corresponding lift history in Fig. 12(b) shows that the peak lift coefficient is observed near the mid stroke which corresponds to a pitch angle of between 55° and 60° and maximum plunge velocity. The cycle averaged aerodynamic coefficients are $C_{L,av} = 0.783$ and $C_{D,av} = 1.417$. Lift in general depends on the structural properties of the wing, wing kinematics, and flow conditions. For an optimal combination of these parameters, a flexible wing can yield more favorable lift than its rigid counterpart\textsuperscript{15}. Furthermore, the pitch angles that correspond to the flexure of the wing are plotted in Fig. 13(a) that have been averaged across three representative cycles. Even after the large initial transients have been damped, there remains some cycle-to-cycle variation in the beam’s response, indicating the response of the flexible wing is not purely periodic. The average passive pitch angle is plotted in blue. Moreover, we include the pitch angle of a honeybee’s wing measured in hover\textsuperscript{43}. The magnitude of the passive pitch compares favorably with the experimental measurements, suggesting that a pure flapping wing motion with passive pitch may explain the wing kinematics of a freely flying honeybee. Effects of flexible wings in flapping flight have been elucidated in the literature, often showcasing an enhanced propulsive force generation while saving the power consumption\textsuperscript{13,15}. How much an insect actively rotates its wing is still an active area of research.

![Figure 11. Time histories of $C_L$ and $C_D$ for 12 flapping periods for the flexible wing with flapping amplitude, $Z = 127^\circ$, $k = 0.124$, $f/f_1 = 0.4$.](image)

![Figure 12. Flexible wing deformation (red) and rigid wing motion (black) using the same resultant wing pitch as the flexible case. $Z = 127^\circ$, $k = 0.124$, $f/f_1 = 0.4$.](image)

2. The necessity of tightly coupling the flight dynamics to the fluid-structure interaction

Prior to discussing the longitudinal response of the system, we must address the sensitivity of the numerical simulation to the integration time step. In Section II.D.4, we discussed our motivation for tightly coupling the equations of motion to the fluid-structure interaction. Another important reason that tight coupling is required in this study is the spurious nature of the solution when larger intervals are used for integrating the equations of motion. In Fig. 14 we demonstrate the effect of averaging the forces prior to applying them to the equations of motion at a sampling frequency less than that of the fluid-structure simulation. Other researchers have also discussed different averaging methods, most notably Orlowski and Girard\textsuperscript{24}, who recommend using quarter-cycle averaging in lieu of full-cycle averaging. Our attempts to apply any kind of averaging other than coupling the equation of motion to the
Figure 13. (a) Resultant wing pitch, $\alpha_{\text{flex,ave}}$, due to wing bending for the flexible wing, plotted with honeybee wing kinematics from Altshuler et al.\textsuperscript{43} (b) Comparison of $C_L$ for flexible and rigid wings during one wingbeat cycle. $Z = 127^\circ$, $k = 0.124, f/f_1 = 0.4$.

A potential cause for the peaks in the aerodynamic forces is that the change in the body displacement from the equation of motion into the Navier-Stokes equation solver increases with the length of the averaging intervals. Conserving the momentum of the fluid under these large body displacements causes significant changes in the pressure distribution around the wing, resulting in the force spikes. This is seen for 20 time steps of the solution in Fig. 15. When the body displacements are provided to the FSI routine every time step, the change in the wing displacement is smooth and therefore the numerical derivatives of that motion are also smooth. If the body displacements are provided less frequently (e.g. every four steps), the additional change in the position of the wing at the fourth step, though small compared to the plunge displacements, causes the velocity and acceleration to exhibit non-physical spikes due to the averaging. These force spikes disappear when the Navier-Stokes equation solver and the equation of motion are coupled at every time step.

On the other hand, a quasi-steady aerodynamics model assumes that the forces and moments depend on the instantaneous wing velocities and accelerations. The relation between the force coefficients and the wing motion is partly based on the linearized aerodynamic theories and in part on empirically obtained parameters. These quasi-steady models neglect history effects, under-representing wing-wake interactions that are typical to hovering insects. As a consequence, a flight dynamics framework coupled to a quasi-steady aerodynamics model may seem to be more robust as it does not account for the flow field at earlier time instances. However, the apparent robustness also suggests that the solution from a quasi-steady model may be artificial, not satisfying the first principles of physics under certain conditions. Assessment of the accuracy of the use of the quasi-steady models in the flight dynamics modeling is one of our future studies.
Figure 15. a) Velocity and b) acceleration of the wing using the following intervals between integrating the equations of motions: 1 step, 4 steps, and 8 steps. The parameter, $h_a$, is the translation of the wing due solely to flapping.

(a) (b)

Figure 16. Longitudinal response of the FWMAV in a near-hover condition. (a) Rates in the body frame where $u^+$ and $w^+$ have units of chords/period and $q^+$ has units of degrees/period; (b) States in the body frame.

3. Longitudinal response of the system with flexible wings

In the rigid wing case, the control inputs selected by Sun and Xiong28 and also considered in this study were the up-stroke and down-stroke pitch angles, and the mid-stroke flapping angle. In the flexible cases, the effective pitch of the wing is a response of the system under the dynamics balance between the wing inertia, elastic restoring force, and fluid dynamic force. Hence, the pitching motion on a flexible wing is no longer available as a control input. We therefore select the wing flapping amplitude ($Z$), the mid-stroke flapping angle ($\xi$), and the stroke plane orientation ($\beta$) as the new control inputs for the flexible case. A point near equilibrium was determined via trial and error, which required inputs of $Z = 127^\circ$, $\xi = 0.2^\circ$, $\beta = 5^\circ$. The resulting response of the system as a function of time is detailed for all three degrees of freedom in Fig. 16. The nondimensional velocities, and displacements in $x$, $z$, and $\theta$ are
plotted for ten wingbeat cycles for both rigid and flexible wing simulations. Length terms are nondimensionalized by the wing chord, and time terms are nondimensionalized by the wing beat period (e.g. $u^+ = u/c_f$).

In order to address the effect of wing flexibility on the system response, we also conduct a rigid-wing simulation for the purpose of comparison. The rigid wing rotates actively about the leading edge. The pitch angles that are assigned to the rigid wing are obtained directly from the flexible wing simulation via a look-up routine. Since the kinematics are so closely aligned, the difference between the flexible wing motion and the rigid wing observed in Fig. 16 arise due to the following physical mechanisms: the presence of camber in the flexible wing, the difference in vortical activity in the flow, and the difference in body motion itself which becomes significant after the sixth period, particularly in the case of body pitch.

The first significant observation seen in Fig. 16 is the wholly different response in some of the degrees of freedom between models with rigid and flexible wings. In the body x-direction, the flexible wing moves aft at a relatively slow rate, and the rigid wing simulations move the body forward at a much higher rate. In the vertical degree of freedom, the response is similar, but it should be noted that the flexible wing creates slightly more lift, which was observed in our previous study as well. The axis is inverted for the vertical direction since positive $z$ is down in the body reference system. With the excess lift, the FWMAV with flexible wings has a higher displacement in the vertical direction. For the pitch degree of freedom, $\theta$ in Fig. 16(b) the growth of the body pitch is much lower for the flexible wing once again. Although both simulations predict the body pitch to decrease, the flexible wing’s response is significantly less than the rigid wing. Most studies in the dynamics and stability of hovering flapping flyers indicate that the flapper is most unstable in pitch. Our results suggest that the instability significantly reduces when a flexible wing is considered.

![Figure 17](http://arc.aiaa.org/doi/10.2514/6.2016-0470)

(a) Rates in the body frame where $u^+$ and $w^+$ have units of chords/period and $q^+$ has units of degrees/period; (b) States in the body frame.

In addition to the hover case, we also tracked the response of both rigid and flexible wing models when the vehicle was subjected to a gust from the x-direction in the body frame of $u = 0.05U$. These results are presented in Fig. 17. For the horizontal body motion, the flexible wing model has a lower response to the perturbation than the rigid wing model. The pitch response is also significantly less for the flexible wing model than the rigid wing model. After ten beat cycles, the rigid wing model reach a nose down pitch of 51.0° whereas the flexible wing model was only 30.7° nose low. The response is the vertical direction is the only degree of freedom where we observe the flexible wing response to be higher than the rigid wing. This is due to the excess lift that is produced by the flexible wing. Once again, we see that allowing for flexible wing motion has a significant effect on the overall response of the body. Although significantly more work is required to quantify and classify these effects, the results...
presented here suggest that omitting the chord-wise flexibility is not appropriate for analysis of many insects and FWMAV designs that have flexible wings.

IV. Concluding Remarks

This paper presents a full nonlinear model of a flapping wing micro air vehicle, as well as three models that we used to solve the flow field and the structural response: rigid wing analysis with a quasi-steady aerodynamic model, rigid wing analysis with a Navier-Stokes solver, and flexible wing with a coupled CFD-CSD solver. All three modeling strategies are tightly coupled with the flight dynamics equations. The longitudinal stability of the system using is assessed using two techniques: a comparison of the prediction of the longitudinal stability about an equilibrium flight condition in hover, and depictions of the longitudinal response of the full nonlinear equations.

The results show that the quasi-steady simulations show reasonably good agreement with the rigid wing Navier-Stokes solutions, and the general response of the bumblebee to perturbations in the longitudinal plane shows the same qualitative behavior. However, a quasi-steady model can yield unphysical solutions as it neglects any history effects. Moreover, the behavior of the bumblebee when modeled with a flexible wing exhibited a different longitudinal response. In particular, the pitching rate of the FWMAV with flexible wings was significantly smaller than the pitching rate with rigid wings given the same starting flight condition in hover. Although more research is required in this area, this study demonstrates that the longitudinal stability of the bumblebee can be significantly affected by chordwise wing flexibility.

Appendices

Appendix A. Addendum to Section II.B.3: Solving the Equations of Motion of Flight Dynamics

The acceleration of the wing’s center of mass, expressed in the body frame, is given by Eq. (A.1)

\[ \ddot{\mathbf{x}}_{\text{cg}} = \ddot{x}_{\text{cg}} + \dot{y}_{\text{cg}} \times \dot{z}_{\text{cg}} + \dot{z}_{\text{cg}} \times \ddot{y}_{\text{cg}} + \dot{w}_{\text{cg}} \times \ddot{w}_{\text{cg}} + \ddot{R}_{\text{hub}} \left( \dot{x}_{\text{cg}} + \dot{y}_{\text{cg}} \times \dot{z}_{\text{cg}} + \dot{z}_{\text{cg}} \times \ddot{y}_{\text{cg}} + \dot{w}_{\text{cg}} \times \ddot{w}_{\text{cg}} \right) \]  

(A.1)

We need to express Eq. (A.1) in a way that isolates those components that have second order derivatives of the state variables, which then becomes Eq. (A.2) where the double underlined terms are those that contain state derivatives and need to be isolated, and the single underlined terms arise from flexible motion of the wing, but are otherwise zero.

\[ \ddot{\mathbf{x}}_{\text{cg}} = \ddot{x}_{\text{cg}} - \left( \ddot{\mathbf{x}}_{\text{cg}} \times \dot{\mathbf{b}}_{\text{cg}} + \left( \ddot{\mathbf{R}}_{\text{hub}} \right) \times \dot{\mathbf{b}}_{\text{cg}} \right) + \ddot{R}_{\text{hub}} \left( \begin{array}{c} \alpha \\dot{\alpha} \\ \gamma \\dot{\gamma} \\ \delta \\dot{\delta} \end{array} \right) \times \ddot{\mathbf{x}}_{\text{cg}} + \ddot{\mathbf{u}}_{\text{cg}} \times \ddot{\mathbf{u}}_{\text{cg}} + \ddot{\mathbf{v}}_{\text{cg}} + \ddot{\mathbf{w}}_{\text{cg}} + \ddot{\mathbf{r}}_{\text{cg}} \right) \]  

(A.2)

The first line of Eq. (A.2) can be rearranged to Eq. (A.3) which allows for insertion into \( H \), which is the mass matrix that must be inverted to isolate \( \dot{\mathbf{v}} \), \( \dot{\mathbf{w}} \), \( \dot{\mathbf{p}} \), \( \dot{\mathbf{q}} \), \( \dot{\mathbf{r}} \) per Eq. (14).

\[ \begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{v}} \\ \dot{\mathbf{w}} \end{bmatrix} = \begin{bmatrix} 0 & -z_{\text{cg}} & y_{\text{cg}} \\ z_{\text{cg}} & 0 & -x_{\text{cg}} \\ -y_{\text{cg}} & x_{\text{cg}} & 0 \end{bmatrix} + \ddot{R}_{\text{hub}} \begin{bmatrix} 0 & -z_{\text{cg}} & y_{\text{cg}} \\ z_{\text{cg}} & 0 & -x_{\text{cg}} \\ -y_{\text{cg}} & x_{\text{cg}} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}} \\ \ddot{\mathbf{v}} \\ \ddot{\mathbf{w}} \end{bmatrix} \]  

(A.3)

Substituting these expanded relationships into the equations of motion results in Eq. (A.4). The first line contains the acceleration terms, and the second line contains the gravitational and aerodynamic forces. The third and fourth lines contain terms that express the states of the body, the prescribed motion of the wing, or the deformation of the wing.
\[
\begin{align*}
\left( m_{\text{body}} + \sum_{i=1}^{\# \text{wings}} m_{w,i} \right) \ddot{\mathbf{v}}_b - \sum_{i=1}^{\# \text{wings}} m_{w,i} \left( \ddot{\mathbf{r}}_{w,i/cg} \right) - \sum_{i=1}^{\# \text{wings}} m_{w,i} \left( R_{w,i-b} \mathbf{\tilde{r}}_{w,i/cg} R_{w,i-b}^T \right) \delta \hat{\mathbf{o}}_b &= -b \mathbf{\tilde{F}}_{\text{aero, body}, b} + m_{\text{body}} b \ddot{\mathbf{g}} + \sum_{i=1}^{\# \text{wings}} \left( b \mathbf{\tilde{F}}_{\text{aero, w}, i} + m_{w,i} b \ddot{\mathbf{g}} \right) - m_{\text{body}} b \delta \hat{\mathbf{o}}_b b \ddot{\mathbf{v}}_b \\
- \sum_{i=1}^{\# \text{wings}} m_{w,i} \left( b \delta \hat{\mathbf{o}}_b b \ddot{\mathbf{v}}_b + b \delta \hat{\mathbf{o}}_b b \delta \hat{\mathbf{o}}_b b \mathbf{\tilde{r}}_{w,i/cg} \right) \\
- \sum_{i=1}^{\# \text{wings}} m_{w,i} R_{w,b} \left( \mathbf{\ddot{v}}_{w,g/o} + 2 w \mathbf{\tilde{v}}_{w} - \mathbf{\tilde{v}}_{w,g/o} - b \mathbf{\tilde{r}}_{w,g/o} \delta \hat{\mathbf{o}}_b + w \delta \hat{\mathbf{o}}_b w \mathbf{\ddot{v}}_{w,g/o} + b \delta \hat{\mathbf{o}}_b b \mathbf{\ddot{r}}_{w,g/o} \right)
\end{align*}
\] (A.4)

In addition to the force equations of motion, the angular equations of motion must undergo the same procedure in order to isolate the highest order derivatives of the state variables, a process that yields Eq. (A.5).

\[
\begin{align*}
\sum_{i=1}^{\# \text{wings}} \left[ b \mathbf{\tilde{M}}_{\text{aero, w}, i} + m_{w,i} b \mathbf{\tilde{r}}_{w,g/o} b \mathbf{\tilde{F}}_{\text{aero, w}, i} \right] &= \sum_{i=1}^{\# \text{wings}} \left[ R_{w,b} \mathbf{\mathbf{\ddot{v}}}_{w,g/o} - m_{w,i} \mathbf{\mathbf{\ddot{v}}}_{w,g/o} \mathbf{\mathbf{\ddot{v}}}_{w,g/o} \right] \\
- \sum_{i=1}^{\# \text{wings}} m_{w,i} b \mathbf{\tilde{r}}_{w,g/o} \left[ b \mathbf{\tilde{M}}_{\text{aero, w}, i} + m_{w,i} b \mathbf{\tilde{r}}_{w,g/o} \right] - b \delta \hat{\mathbf{o}}_b b \mathbf{\tilde{L}}_{\text{b,cg}, b} \delta \hat{\mathbf{o}}_b \\
- \sum_{i=1}^{\# \text{wings}} \left[ R_{w,b} \mathbf{\mathbf{\tilde{L}}}_{w,o} + \mathbf{\tilde{L}}_{w,o} \right] \left( w \mathbf{\tilde{v}}_{w,g/o} - \mathbf{\tilde{v}}_{w,g/o} \right) + w \delta \hat{\mathbf{o}}_b w \mathbf{\tilde{L}}_{w,o} \delta \hat{\mathbf{o}}_b \\
- \sum_{i=1}^{\# \text{wings}} \left( R_{w,b} \mathbf{\mathbf{\tilde{r}}}_{w,g/o} - w \mathbf{\tilde{r}}_{w,g/o} \right) \left[ b \delta \hat{\mathbf{o}}_b b \mathbf{\tilde{v}}_{w,g/o} + b \delta \hat{\mathbf{o}}_b b \delta \hat{\mathbf{o}}_b b \mathbf{\tilde{r}}_{w,g/o} \right] \\
- \sum_{i=1}^{\# \text{wings}} m_{w,i} b \mathbf{\tilde{r}}_{w,g/o} \left[ -w \mathbf{\tilde{r}}_{w,g/o} b \mathbf{\tilde{v}}_{w,g/o} + b \delta \hat{\mathbf{o}}_b b \mathbf{\tilde{r}}_{w,g/o} b \mathbf{\tilde{r}}_{w,g/o} \right] \\
- \sum_{i=1}^{\# \text{wings}} m_{w,i} b \mathbf{\tilde{r}}_{w,g/o} \left[ b \delta \hat{\mathbf{o}}_b b \mathbf{\tilde{v}}_{w,g/o} + b \delta \hat{\mathbf{o}}_b b \delta \hat{\mathbf{o}}_b b \mathbf{\tilde{r}}_{w,g/o} \right]
\end{align*}
\] (A.5)

Equations (A.4)-(A.5) represent the full equations of motion for the multibody model of the bumblebee for the linear and angular force-acceleration balance. In these equations, the tilde over a vector quantity indicates the matrix form of the cross product is used. Although the terms are grouped differently, they are the same relations presented by Wu, Zhang, and Sun30. In this form, Eq. (A.4)-(A.5) can be expressed as Eq. (A.6) where \( H \) is a 6x6 matrix, defined by Eq. (A.7) and the right hand sides (RHS) of Eqs. (A.4)-(A.5) are not repeated in Eq. (A.6).

\[
\begin{bmatrix}
\dot{u} \\
\dot{v} \\
\dot{w} \\
\dot{\rho} \\
\dot{q} \\
\dot{r}
\end{bmatrix} = [H]^{-1} \begin{bmatrix}
\text{RHS of Eq. A.32} \\
\text{RHS of Eq. A.33}
\end{bmatrix}
\] (A.6)
Appendix B. Computational Setup and Grid and Time Step Sensitivity Study

In order to assess the effects of grid resolution and time step size at the bumblebee scale at $Re=1\times10^3$, grid and time step convergence studies were conducted to determine the optimal grid resolution, domain size, and time step.

A structured grid around a 2% thick flat plate with unit chord length was used. A motion based on Xiong and Sun\textsuperscript{28} was prescribed on the rigid wing. A no-slip boundary condition was prescribed on the flat plate surface, and outer boundaries of the domain were set as inlet. A rectangular domain showed a better convergence rate and fewer velocity-pressure coupling iterations per time step than a circular domain. After assessing the effects of domain shape and domain size, we found that a rectangular domain with outer boundary at 50 chords lengths in the direction normal to the wing had an optimal distribution (see Fig. B.1). Lift histories for each grid are provided in Fig. B.2.

\begin{equation}
[H] = \begin{bmatrix}
m_{\text{tot}} [I_{3\times3}] \\
\sum_{i=1}^{s \text{wings}} m_{w,i} \left( \tilde{r}_{w,i}^{cg,j} + R_{w,i}^{al,k} \tilde{r}_{w,i}^{al,k} R^T_{w,i} \right) \\
- \sum_{i=1}^{s \text{wings}} m_{w,i} \left( \tilde{r}_{w,i}^{cg,j} + R_{w,i}^{al,k} \tilde{r}_{w,i}^{al,k} R^T_{w,i} \right)
\end{bmatrix}
\end{equation}

Figure B.1. Overview of the computational mesh for 61×101 grid. (a) Computational domain and boundary conditions. (b) Zoomed in view of the mesh around the flat plate shows the rectangular computational domain around the flat plate.

Four levels of grids were created ranging from a coarse to extra-fine grid with resolutions 31×51, 61×101, 121×201 and 241×401 respectively. In comparison, Xiong and Sun\textsuperscript{28} used a three dimensional O-H type computational grid with the outer boundary fixed at 20 chord lengths from the wing and 71×73×96 points in the normal direction, along the body axis and in the azimuthal direction, respectively. Convergence was assessed systematically based on the error norms of lift coefficient $C_L$ over two motion cycles with the solution on the 241×401 grid considered as an estimate of the exact solution. The $L_1$ and $L_2$ norms were calculated with 480 time steps per motion cycle per Eq. (B.1), where $k$ indicates the mesh level and $N$ is 480, the total number of time steps over which the norm is calculated. Based on the $L_1$ and $L_2$, tabulated in Table B.1, we chose 121×201 grid for all Navier-Stokes equation computations.
Figure B.2. The lift history over one motion cycle for four grid levels shown in Table B.1.

\[ L_1 \text{norm}_k = \sum_{n=1}^{N} |f_{k,n} - f_{\text{exact},n}| / N \]

\[ L_2 \text{norm}_k = \left( \sum_{n=1}^{N} |f_{k,n} - f_{\text{exact},n}|^2 / N \right)^{\frac{1}{2}} \] (B.1)

Time sensitivity was also performed in the same manner on the 121×201 grid. Four levels of time steps per motion cycle were chosen: 240, 480, 960, and 1920. The criteria for assessing temporal convergence was based on the \( L_1 \) and \( L_2 \) norms of lift coefficient \( C_L \) with solution at 1920 time steps per motion cycle considered as the most accurate solution. Based on the convergence seen in Table B.1, 960 time steps per flapping cycle were considered for all computations.

| Table B.1. Spatial and temporal sensitivity study at bumblebee scale. |
|-------------------|---------------------|-------------------|
|                   | \( L_1 \) Norm, \( C_L \) | \( L_2 \) Norm, \( C_L \) |
| Spatial 31×51     | 0.060               | 0.085             |
| 61×101            | 0.034               | 0.051             |
| 121×201           | 0.022               | 0.035             |
| 241×401 (baseline)|                    |                   |
| Temporal (time steps) 240 | 0.496               | 3.90              |
| 480               | 0.311               | 2.57              |
| 960               | 0.152               | 1.40              |
| 1920 (baseline)   |                    |                   |
References


